REPORT No. 142

GENERAL THEORY OF THIN WING SECTIONS

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RÉSUMÉ.

The following paper contains a new, simple method of calculating the air forces to which thin wings are subjected at small angles of attack, if their curvature is not too great. Two simple integrals are the result. They contain only the coordinates of the wing section. The first integral gives the angle of attack at which the lift of the wing is zero, the second integral gives the moment experienced by the wing when its angle is zero. The two constants thus obtained are sufficient to determine the lift and moment for any other angle of attack. This refers primarily to a two-dimensional flow in a nonviscous fluid. However, in combination with the theory of the aerodynamical induction, and with our empirical knowledge of the drag due to friction, the results are valuable for actual wings also. A particular result obtained is the calculation of the elevator effect. The following is an outline of the subject as treated in this report:

I. Introduction.
II. Calculation of the elevator effect.
III. General formula for any section.
IV. Examples of the zero angle.
V. Thin sections with upper and lower boundaries.
VI. The moment coefficient.
VII. Examples of the moment coefficient.
VIII. Table of the sections investigated.

I. INTRODUCTION.

By changing the angle between the stabilizer and the elevator the wing section formed by the combination of stabilizer and elevator is altered, and this alteration gives rise to new aerodynamical forces. It is useful to discuss this phenomenon from the theoretical point of view, however imperfect the result may be as a consequence of neglecting the viscosity of the air. A theoretical investigation at least gives the limit of what to expect. It enables the investigator to survey and keep in mind a great number of isolated experiences, whether the agreement between theory and experience be more or less close. It induces him to reflect on the phenomenon and thus becomes a source of progress by guiding him to new observations and experiments. It has often occurred even that some relation was thought to be confirmed by experience till the progress of theory made the relation improbable. And only then the experiments confirmed the improved relation, contrary to what they were supposed to do before. A very conspicuous example of this is the discovery of differences in the atomic weight of certain elements. But is it really necessary to plead for the usefulness of theoretical work? This is nothing but systematical thinking and is not useless as sometimes supposed, but the difficulty of theoretical investigation makes many people dislike it.

In this first section I wish to give a short summary of the theory which I am going later to apply and to expand. This theory deals with the relation between the shape of a wing section and the air forces applied to it by a nonviscous fluid. Only the two-dimensional problem is considered. The theory thus forms the completion of the theory of the induced drag, in which latter the three-dimensional arrangement of the wings and the lift produced by them alone is considered, without paying attention to the details of producing the lift. The value of the induced drag and the effective angle of attack of every part of the wings result from the calculation. The theory of the wing section, however, gives no drag at all, for the drag additional to the induced drag is due to viscosity. Nor does the theory of the wing section give the true value of the maximum lift. It can be stated, therefore, that the theory of the wing section in its present state gives no indication whatsoever of the practical value of the wing investigated. Still there remain three important pieces of information which can be derived from the theory, all more or less agreeing with the real phenomenon. These are the relation between the angle of attack and the lift, in particular the angle of attack for zero lift, the travel of the center of pressure, and the distribution of pressure. It has to be kept in mind that the angle of attack thus calculated for a particular lift coefficient is not yet the true angle of attack of a finite wing. The induced angle of attack has to be added.

We are indebted for the theory of the wing section to Kutta. He showed how the method of the two-dimensional potential can be used to calculate the flow around wing sections and hence to deduce the resulting air forces. He confined himself to the straight line and simple circular segments. His idea is to pick out among the multitude of possible potential flows that particular one around the wing section, which at great distance degenerates into parallel flow and which leaves the wing section at the rear edge. His results are simple and important. The direction of the air flow in the case of zero lift of a circular segment of small curvature is parallel to the line dividing into equal parts the angle between the chord and the tangent at the rear end. The lift is proportional to the sine of the angle of attack. The slope of the curve of the lift coefficient plotted against the angle of attack is almost independent of the shape and is 2π (the angle being measured in arc and the lift coefficient being formed by dividing the lift per unit of area by the dynamical pressure). That is, for small lift:

$$L=2\pi\,S\sin\alpha_1 V^2\frac{\rho}{2}$$

Joukowsky extended the theory, and investigated sections which at their rear end almost coincide with a circular segment, having there a common tangent for the upper and lower side. The entire form is generated from the circle, a circular segment forming as it were the skeleton of a Joukowsky section. Considering the connecting line between the rear edge and a pole near the center of curvature of the leading edge as the theoretical chord, the rule for the direction of zero lift remains as before. The slope of the lift curve is hardly changed; the lift is proportional to the sine of the angles as before.

Karman replaced the circular segment in the Joukowsky section by one formed by two circular segments. This is already mentioned in the second paper of Kutta. These sections have two different tangents at the rear end, and the line which divides the rear angle into two equal parts determines the direction of zero lift together with the theoretical chord as before. The law for the lift is the same again as for the circular segments of Kutta. Mises discusses in a general way how to obtain even more general sections and proves some general theorems concerning them. The most important is the theorem that the slope of the lift curve plotted as before is never smaller than 2π, and is always exactly 2π if the section is thin and the curvature small. So far it can be stated that only sections are investigated, the medial line of which is a circular segment. If the section is only moderately thick and if the curvature is moderate, too, the lift agrees with that of the segment according to the law found by Kutta.

II. CALCULATION OF THE ELEVATOR EFFECT.

In this paper I intend to investigate any thin section of small curvature at small angles of attack. It is necessary to discuss first more closely the method used by Kutta for the calculation of the lift of a wing section. He starts with an entire circle, and considers the potential flow around it, which was well known before him. It can be obtained by superposition of the symmetrical flow around the circle and a vortex in the center of the circle. By changing the strength of the vortex the point where the flow leaves the circle can be chosen at will for any given direction of the flow at a great distance. The lift produced by the flow is proportional to the product of the velocity at a great distance and the strength of the vortex. It is not essential for the calculation that the described flow never really occurs; it is only a means of calculation. Kutta transforms now the plane in such a way that it remains unaltered at a great distance from the circle. The circle itself, however, is transformed into a new curve, the wing section to be investigated. The transformation has to be of that kind which is called isogonal and leaves in general all angles unaltered. It is well known that each analytical function gives such a transformation, the plane represented by complex numbers. The rear edge of the new section corresponds to one particular point of the circle. After having found this point it is only necessary to determine the lift of that flow around the circle that leaves the circle at that particular point and at a great distance has the same magnitude and direction as the flow around the section investigated. This lift equals the lift experienced by the section.

The simplest case, the one, moreover, which I need in the following development, is the straight line. The transformation of the circle with the radius 1 and its center coinciding with the origin of the system of coordinates into the straight line connecting the two points $\zeta=-2$ and $\zeta=+2$ is expressed by the analytical function

$$\zeta=z+\frac{1}{z} \tag{1}$$

For large values of z the function degenerates into $\zeta=z$ and hence leaves the plane unaltered at a great distance. The rear edge of the straight line $\zeta=+2$ corresponds to the point $z=+1$ of the circle. Each point of the straight line corresponds to those two points of the circle which have half the abscissa. It is known now that the lift of the circle for the flow which leaves it at the point $z=+1$ and whose direction at a great distance has the angle α with the real axis is

$$8\pi V^2\frac{\rho}{2}\sin\alpha$$

for the unit length of the cylinder. More generally the lift is r times as great if r is the radius instead of 1. The lift of the straight line, or the rectangular plate represented by it, is the same, and the lift coefficient therefore, since the chord has the length 4, is $2\pi\sin\alpha$.

It is not necessary for the following development to enter into the details of the flow around Kutta segments, or Joukowsky and Karman sections. I at once proceed to the subject of this paper. In his paper Karman speaks of the possibility of finding the transformation for any section approximately, if this section differs but slightly from another section the transformation of which is known. He gives also the formulas for the approximation, but he does not prove them. I proceed to apply a method obviously similar to that of Karman. The formulas I obtain, however, do not agree with those given by him. I am going to study the effect of an infinitesimal change of a section, and I chose as the original section the straight line. I begin with the investigation of a broken line, the two portions of which form almost 180°. This broken line represents a tail plane, the elevator being slightly turned from the mean position. The length of the tail plane is 4, the two ends coincide with the ends of the original straight line at the points $\zeta=\pm2$. This is necessary, the function of transformation being unsteady in these two points. The lift produced by the small change of the shape is small, too, of course, but the ratio of the effect to the change which causes it is finite and can be calculated.

Imagine the straight line and the circle drawn in Figure 1 and above the straight line the new section consisting of two straight lines. The ordinates of the new section may be called

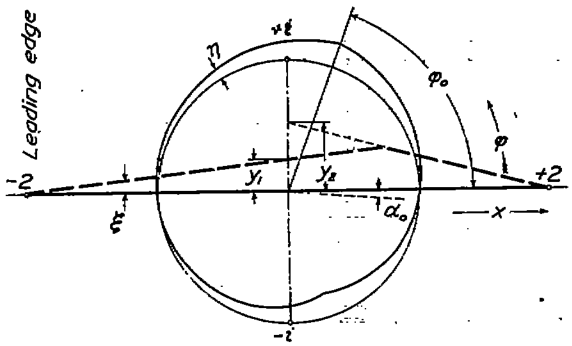


FIG. 1. Transformation of the tail plane.

ξ and the abscissæ x. If y_1 and y_2 denote the ordinates of the intersection points of the two lines of the new section with the imaginary axis, we have:

$$\xi = y_2 \frac{2-x}{2} \text{ for the one part and}$$

$$\xi = y_1 \frac{2+x}{2} \text{ for the other.}$$

I try now to find the curve in the neighborhood of the circle which corresponds to the new section according to the transformation:

$$\zeta = z + \frac{1}{z}. \tag{1}$$

The derivative of ζ with reference to z is:

$$\frac{d\zeta}{dz} = 1 - \frac{1}{z^2}. \tag{2}$$

Introducing the angle φ between the real axis and the radius at each point of the circle, the points of the circle are represented by—

$$z = e^{i\varphi} \tag{3}$$

and (2) can be written—

$$\frac{d\zeta}{dz} = 1 - e^{-2i\varphi} = 2 \sin \varphi \, i \, e^{-i\varphi} \tag{4}$$

$d\zeta$ is the change of the originally straight tail plane and equals i ξ, that is to say is:

$$d\zeta = i\, y_2 \,(1 - \cos \varphi) \tag{5}$$

on the right side of the hinge and

$$d\zeta = i\, y_1 \,(1 + \cos \varphi) \tag{6}$$

on the left side of it, the positive real axis being supposed to be drawn toward the right from the hinge.

At the hinge (5) and (6) agree, the angle corresponding to the abcissa of the hinge being denoted by φ_1, it follows therefore that—

$$y_2\,(1 - \cos \varphi_0) = y_1\,(1 + \cos \varphi_0)$$

$$y_1 = y_2 \frac{1 - \cos \varphi_0}{1 + \cos \varphi_0}. \tag{7}$$

By substituting (5) resp. (6) in (4) I obtain

$$dz=\frac{d\zeta}{1-e^{-2\varphi i}}=\frac{iy_2(1-\cos\varphi)}{1-e^{-2\varphi i}}\text{ resp. }i\,y_1\frac{(1+\cos\varphi)}{1-e^{-2\varphi i}} \tag{8}$$

and the radial small distance of the new curve from the original circle is

$$\eta=\frac{y_2}{2}\frac{1-\cos\varphi}{\sin\varphi}=\frac{y_2}{2}\tan\frac{\varphi}{2}\text{ between }\varphi=\begin{matrix}-\varphi_0\\+\varphi_0\end{matrix} \tag{9}$$

$$\eta=\frac{y_1}{2}\frac{1+\cos\varphi}{\sin\varphi}=\frac{y_1}{2}\cot\frac{\varphi}{2}\text{ between }\varphi=\begin{matrix}+\varphi_0\\-\varphi_0\end{matrix} \tag{10}$$

The problem is now to transform the original circle $z=e^{\varphi i}$ into the curve $e^{\varphi i}\,(1+\eta)$ so that the value at infinity remains unaltered.

For the present it is not necessary really to perform this transformation; it is sufficient to imagine the transformation performed approximately by the function:

$$\frac{1}{Z_1}=\frac{1}{Z}\left(1+\frac{a_1}{Z}+\frac{a_2}{Z^2}+\frac{a_3}{Z^3}+\cdots\right) \tag{11}$$

where the a's are coefficients to be determined properly. This transformation indeed leaves the plane at infinity unaltered. It is exact if η approaches zero. For the coefficients are imagined to be determined by the following method. Let η be developed in a Fourier's series between 0 and 2π.

$$\eta=\frac{1}{2}A_0\begin{matrix}+A_1\cos\varphi+A_2\cos2\varphi+A_3\cos3\varphi++\\+B_1\sin\varphi+B_2\sin2\varphi+B_3\sin3\varphi++\end{matrix} \tag{12}$$

As is well known, the coefficients A and B are the integrals:

$$\begin{aligned}A_n&=\frac{1}{\pi}\int_0^{2\pi}\eta\cos n\varphi\,d\varphi\\B_n&=\frac{1}{\pi}\int_0^{2\pi}\eta\sin n\varphi\,d\varphi\end{aligned} \tag{13}$$

If, as in this case, the section has no thickness, the coefficients A are all identically zero. The coefficients a in (11) may be formed according to the equations:

$$a_1=-(A_1+iB_1)$$

$$a_n=-(A_n+iB_n)$$

This transformation does not give the desired transformation exactly. The point of the circle $z=e^{\varphi i}$ is transformed into the point.

$$\frac{1}{Z_1}=e^{-\varphi i}\begin{Bmatrix}1-A_1\cos\varphi-B_1\sin\varphi-A_2\cos2\varphi-B_2\sin2\varphi--\\+i(A_12\varphi-B_1\cos\varphi+A_2\sin2\varphi-B_2\cos2\varphi+-\end{Bmatrix}$$

$$=e^{-\varphi i}[1-\eta+i\,(A_1\sin\varphi+A_2\sin2\varphi+)-i\,(B_1\cos\varphi+B_2\cos2\varphi+)] \tag{14}$$

η being supposed to be very small, $\frac{1}{z_1}=e^{-\varphi i}\,(1-\eta)$, the value desired. Hence the point represented by (14) does not exactly coincide with the new curve in the neighborhood of the circle, but differs from it by the last two brackets of (14). The difference is, however, small when compared with η. For the additional vector is parallel to the circle, so is approximately the η-curve. The end of the vector z is therefore situated almost at the curve, but for a different radius vector than that of the original point of the circle z. (14) can be considered as the desired function of transformation. Since for $\varphi=0$, $\eta=0$, the curve which by the transformation $\zeta=z+\frac{1}{z}$ is transformed into the modified tail surface coincides with the circumference of the original circle at its two ends. The problem then is to see by how much the point of the circle $z=1$ at the end of the diameter is displaced by the transformation (1). This may be calculated easily, as will be shown. When deduced it will give the value of the additional vector, which we may assume can be used for other points in the neighborhood, too, because it is a continuous quantity. So, the point of the circle corresponding to $\varphi=0$ for the transformed curve is displaced an amount equal and opposite to this.

The position $e^{\varphi_1 i}$ of the point of the new curve corresponding to the point $z=1$ of the circle is found by using equation (14) and substituting $\varphi=0$. η is zero at that point, the changed section coinciding at the ends with the original section. All sines are zero and there remains:

$$e^{-\varphi i}=1-i\,(B_1+B_2+B_3++\;\cdot\;\cdot\;\cdot\;\cdot \tag{15}$$

The right-hand side of (15) is approximately equal to $e^{-(B_1+B_2+)}$ Hence

$$\varphi_1=B_1+B_2+B_3++\;\cdot\;\cdot\;\cdot\;\cdot \tag{16}$$

It would be possible to determine the B's and to find the value of φ_1 by adding them. This procedure, however, can be simplified. By going back to equation (13), (16) can be written:

$$\varphi_1=\frac{1}{\pi}\int_0^{2\pi}\eta\;(\sin\varphi+\sin 2\varphi+\sin 3\varphi++)\;d\varphi. \tag{17}$$

The bracket in (17) is formally the development of $\cot\frac{1}{2}\varphi$ in a Fourier's series, though indeed this series is not convergent. For

$$\frac{1}{2\pi}\int_0^{\pi}\cot\frac{\varphi}{2}\sin n\,\varphi\,d\,\varphi=\frac{1}{2\pi}\int_0^{\pi}\frac{\cos\varphi/2}{\sin\varphi/2}\,2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\,\frac{\sin n\varphi}{\sin\varphi}\,d\varphi \tag{18}$$

$$=\frac{1}{2\pi}\int_0^{\pi}(\cos\varphi+1)\Big[(2\cos(n-1)\,\varphi+2\cos(n-3)\varphi++\Big]d\varphi \text{ if } n \text{ is odd.}$$

or

$$=\frac{1}{2\pi}\int_0^{\pi}(\cos\varphi+1)\Big[(2\cos(n-1)\,\varphi+2\cos(n-3)\,\varphi+++1\Big]d\varphi \text{ if } n \text{ is even.}$$

This transformation of $\frac{\sin n\varphi}{\sin\varphi}$ results the most simply if we go back to the equation

$$\sin n\varphi=\frac{1}{2i}\;(e^{n\varphi i}-e^{-n\varphi i})$$

and divide this by

$$\sin\varphi=\frac{1}{2i}\,(e^{\varphi i}-e^{-\varphi i})\;.$$

The result

$$e^{(n-1)\varphi i}+e^{(n-3)\varphi i}+e^{(n-5)\varphi i}+\;.\;.\;.$$

is identical with

$$2\cos(n-1)\,\varphi+2\cos(n-3)\,\varphi++\qquad\qquad+1 \text{ if } n \text{ is even.}$$

In both cases, whether n be odd or even, (18) equals $\frac{1}{2}$. And although the bracket in (17) is not convergent, the integral (17) is convergent in general and can be written

$$\varphi_1=\frac{1}{2\pi}\int_0^{2\pi}\eta\cot\frac{\varphi}{2}\,d\varphi \tag{19}$$

(17) represented as we remember the angle of the radius vector through the point of the curve corresponding to the point $z=1$, $\varphi=0$ of the circle. According to what has been said, the point of the circle corresponding to point $\varphi=0$ of the curve has the radius vector

$$\alpha_0 = -\frac{1}{2\pi}\int_0^{2\pi} \eta \cot \frac{\varphi}{2}\, d\varphi \tag{20}$$

I have thus obtained the point of the circle corresponding to the rear edge of the curve investigated. The lift of the section equals the lift of the circle provided that the potential flow leaves it at the point determinated. The lift is zero, in particular if the circulation vanishes and the fluid flows symmetrically around the circle. It is easy to see that in that case the velocity at infinity is parallel to the radius vector of the point determined. Hence it appears that the α_0 just found is the angle of attack for the section investigated necessary for zero lift, i. e. is the "angle of zero lift."

I at once proceed to calculate this integral for the present problem. η has to be substituted from equation (10) and (11). Hence the integral for α_0 consists of two parts:

$$-\alpha_0 = \frac{1}{2\pi}\int_{-\varphi_0}^{+\varphi_0} \frac{y_2}{2}\, d\,\varphi + \frac{1}{2\pi}\int_{+\varphi_0}^{-\varphi_0} \frac{y_1}{2} \cot^2 \frac{\varphi}{2}\, d\varphi \tag{21}$$

The solution is

$$-\alpha_0 = \frac{1}{2\pi}\,\frac{y_1}{2}\left|\, 2 \cot \frac{\varphi}{2} + \varphi \right|_{-\varphi_0}^{+\varphi_0} + \frac{1}{2\pi}\,\frac{y_2}{2}\left|\, \varphi \right|_{-\varphi_0}^{+\varphi_0}. \tag{22}$$

Substituting (7) for y_1 I obtain finally

$$-\alpha_0 = \frac{y_1}{2\pi}\left(2 \tan \frac{\varphi_0}{2} + \frac{2\varphi_0}{1+\cos \varphi_0} - \frac{\pi\,(1-\cos \varphi_0)}{1+\cos \varphi_0}\right) \tag{23}$$

For positive y_1, that is, for the elevator bent down, α_0 is negative. The flow around the circle being horizontal at a great distance leaves the circle below the horizontal axis. Such an elevator with the line of connection of the two ends of the whole tail plane horizontal experiences a positive lift therefore according to the result of the development.

At the same time the stabilizer is no longer horizontal and there remains an elementary calculation in order to obtain the effect of the elevator separately. This is especially simple if elevator and stabilizer are equal and hence $\varphi_0 = 90°$. Then (23) gives $\alpha_0 = -\frac{y_1}{\pi}$. The angle between stabilizer and elevator is y_1 (the length of the whole tail plane being 4) and the angle of attack of the stabilizer is $-\frac{1}{2}y_1$. The positive turning of the elevator not only neutralizes the effect of the negative angle of attack of the stabilizer, but there is also the effect of the angle of attack y_1/π in addition. Hence the whole effect is $y_1\,(\frac{1}{2}+1/\pi)$ and per unit of angle of elevator to stabilizer divided by the ratio of the elevator to the whole tail plane it is:

$$\frac{y_1\left(\frac{1}{2}+\frac{1}{\pi}\right)}{\frac{1}{2}\,y_1} = 1 + \frac{2}{\pi} = 1.64$$

For other positions of the hinge the ratio of the length of the elevator to that of the tail plane is $\frac{1}{2}\,(1-\cos \varphi_0)$, the angle between stabilizer and elevator is:

$$y_1\left(\frac{1-\cos \varphi_0}{2\,(1+\cos \varphi_0)} + \frac{1}{2}\right)$$

and the angle of attack of the stabilizer is

$$\frac{y_1\,(1-\cos \varphi_0)}{2\;(1+\cos \varphi_0)}$$

Let K be the coefficient which gives the effect of the elevator turning according to the equation:

$$\alpha \text{ effective} = \frac{E}{E+S} K\beta$$

α effective is the angle of attack of the whole undeformed tail plane which has the same effect as the turning of the elevator by the angle β; E denotes the chord of the elevator and S the chord of the stabilizer. Then K is

$$K = \frac{\alpha_0 + y_1 \dfrac{1-\cos\varphi_0}{2(1+\cos\varphi_0)}}{\dfrac{1}{2}\left(1-\cos\varphi_0\right) y_1 \left(\dfrac{1-\cos\varphi_0}{2(1+\cos\varphi_0)} + \dfrac{1}{2}\right)}$$

Substituting α_0 from equation (20), this can be written

$$K = \frac{2}{\pi}\frac{\varphi_0 + \sin\varphi_0}{1-\cos\varphi_0}$$

In Figure 2 the coefficient obtained is plotted against the elevator ratio $\frac{E}{E+S} = \frac{E}{T}$. If the tail plane consists only of the elevator the coefficient of course is 1 and the calculation gives

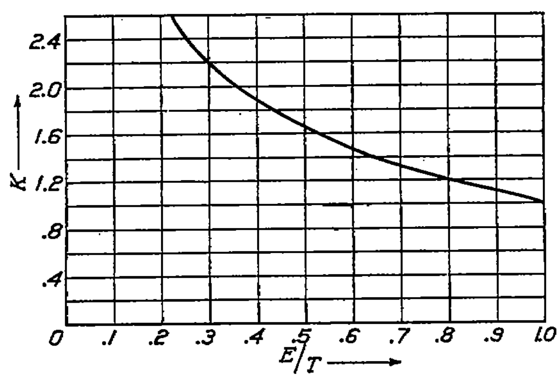


FIG. 2. Elevator effect "K" plotted against elevator/tail plane.

this. The tests with flat tail plane models show considerably smaller values of K, about 1.3. The entirely flat section is aerodynamically so bad that 40 per cent of the elevator effect is lost. A test with a better shaped section, however, gave $K = 1.7$, which agrees better with the result of the theoretical calculation.

III. GENERAL FORMULA FOR ANY SECTION.

The previous calculation of the angle of attack for zero lift shows that this angle is obtained by integrating an expression which contains the coordinates of the central curve of the section investigated as a factor. Hence the angle for zero lift is a linear function of the coordinates, and the law of superposition holds true. That is to say, the zero angle of any section is the sum of the zero angles of any other sections if the coordinates of the first sections are also the sum of the coordinates of the other sections at each point. The result of the previous calculation hence holds true for thin tail planes also, the central curve of which originally was not a straight line but was slightly curved in any way. And the result can be used also for the calculation of the effect of two elevators in front and at the end of the stabilizer turned by any small angle. If $\frac{E_1}{T}$ denotes the ratio of the front elevator of the entire tail plane, the factor K' is to be taken from Figure 2 for $\frac{E}{T} = \left(1 - \frac{E_1}{T}\right)$ and the factor K_1 of the front tail plane then is $K_1 = K' \frac{T - E_1}{E_1} - 1$. This results by superposition of the two tail planes, each with one hinge only.

By increasing the number of the hinges, at last every shape of the section can be obtained. It is more convenient, however, to go back to the second term of equation (21), which represents the part of the zero angle due to the stabilizer. Imagine any curve to be composed of a zigzag line, consisting of an infinite number of pieces of radii vectors originating in the leading edge of the section and of connecting short vertical lines. (Fig. 3.) These vertical lines do

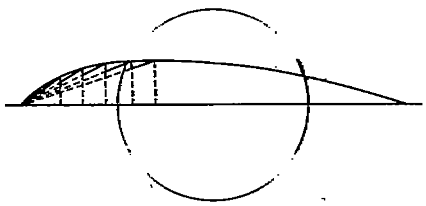


FIG. 3. Zig-zag line replacing the curve of section.

not furnish a contribution to the integral of the zero angle; and this integral is obtained by substituting in

(21a)
$$-\alpha_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{y_2}{2}d\varphi$$

the value of y_2 for each point. This is, if ξ denotes the ordinate of the section

(24)
$$y_2=\frac{\xi}{1-\cos\varphi}$$

hence

(25)
$$-\alpha_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{\xi}{2}\,\frac{1}{1-\cos\varphi}d\varphi$$

Eliminating finally the angle φ and introducing instead of it the position x of the point of the chord, $x=\pm 2$ being the ends of the chord,

$$\cos\varphi=\frac{1}{2}x$$

$$d\varphi=-\frac{1}{2}\frac{dx}{\sin\varphi}=-\frac{1}{2}\frac{dx}{\left[1-\frac{x^2}{2}\right]^{\frac{1}{2}}}$$

I obtain

(26)
$$-\alpha_0=\frac{1}{4\pi}\int_{-2}^{+2}\xi\frac{dx}{\left(1-\frac{x}{2}\right)^{\frac{3}{2}}\left(1-\frac{x}{2}\right)^{\frac{1}{2}}}$$

or, for the length 2 of the chord,

(27)
$$\alpha_0=-\frac{1}{\pi}\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}}$$

In the general case that the length of the chord is t the angle is

(27a)
$$\alpha_0=-\frac{4}{t^2\pi}\int_{-\frac{t}{2}}^{+\frac{t}{2}}\frac{\xi dx}{\left(1-\frac{2x}{t}\right)\sqrt{1-\left(\frac{2x}{t}\right)^2}}$$

The integral (27a) is the essential result of the previous development. The problem I am dealing with is the calculation of the theoretical lift of a line-shaped section of small curvature.

For this calculation I first deduce the angle of attack which gives the lift zero. I choose the scale so that the chord is 2. On this curve with the ordinates ξ from the chord and the rear edge at the point $x=1$ of the chord, the other end having the abcissa $x=-1$, I apply the integral (27). The angle of zero lift toward the chord results. Omitting for the moment the induced angle of attack, the theoretical lift is then

$$L=2\,\pi\,V^2\frac{\rho}{2}\sin\,(\alpha-\alpha_0)\;S$$

It is exact enough to replace the sine by the angle. Introducing now the induced angle of attack

$$\alpha_i=\frac{L}{\pi b^2\,V^2\frac{\rho}{2}}\;57.3\text{ in degrees.}$$

while b is the span of the monoplane, the angle of attack corresponding to any lift is

$$\frac{\alpha}{57.3^\circ}=-\frac{4}{t^2\pi}\int_{-\frac{t}{2}}^{+\frac{t}{2}}\frac{\xi dx}{\left(1-\frac{2x}{t}\right)\sqrt{\left(1-\left(\frac{2x}{t}\right)^2\right)}}+\frac{1}{\pi}\frac{L}{b^2V^2\frac{\rho}{2}}+\frac{1}{2\pi}\frac{L}{S\,V^2\frac{\rho}{2}}$$

IV. EXAMPLES OF THE ZERO ANGLE.

In this part I proceed to calculate some examples of the zero angle. It is sufficient to calculate some typical curves; any more complicated one can be created by superposing of these.

(a) The section may be represented by the curve

$$\xi=y(1-x^2)$$

where y is a small quantity and denotes the greatest ordinate. This curve is typical of simple symmetrical arcs, like a circular segment of practically constant curvature. The integral (27) is now

$$\alpha_0=-\frac{1}{\pi}y\int_{-1}^{+1}\frac{1-x^2}{(1+x)\sqrt{1-x^2}}dx$$

$$=-\frac{1}{\pi}y\int_{-1}^{+1}\sqrt{\frac{1-x}{1+x}}dx$$

$$\alpha_0=-\frac{y}{\pi}\Big|\text{arc.}\sin x+\sqrt{1-x^2}\Big|_{-1}^{+1}$$

$$\alpha_0=-y.$$

The angle between the tangent at the rear edge and the chord is $-2y$. In accordance with the results of Kutta, Joukowsky, and Karman the direction of the zero flow is that of the line which divides into two equal halves the angle between the rear tangent and the chord.

(b) The example of the section consisting of two equal straight lines is contained in the second part of this paper. This section lies above the preceding at all points if the end tangents coincide and it is to be expected therefore that the zero angle is greater. It was found to be $\frac{2}{\pi}$ of the rear angle between the elevator and the chord.

(c) Unsymmetrical sections consisting of two or three straight lines are contained in section II of this paper.

(d) A continual unsymmetrical curve can be obtained for instance by the expression

$$\xi=y(1+x)^n(1-x)^m$$

For $m=n=1$ the segment discussed first results. The zero angle is

$$\alpha_0=-\frac{y}{\pi}\int_{-1}^{+1}(1+x)^{n-\frac{1}{2}}(1-x)^{m-3/2}dx$$

The integration is especially simple if $m=3/2$. Then I obtain

$$\alpha_0=-\frac{y}{\pi}\int_{-1}^{+1}(1-x)^{n-\frac{1}{2}}dx$$

the integral is

$$\alpha_0=-\frac{y}{\pi}\left|\frac{(1-x)^{n+\frac{1}{2}}}{n+\frac{1}{2}}\right|_{-1}^{+1}$$

and the angle has the value

$$\alpha_0=-\frac{y}{\pi}\frac{2^{n+\frac{1}{2}}}{n+\frac{1}{2}}$$

The maximum of the section has the abscissa $\dfrac{n-\frac{3}{2}}{n+\frac{3}{2}}$, or, otherwise expressed, its distance from the leading edge is $\dfrac{2n}{2n+3}$ of the chord. For the symmetrical case $n=3/2$ the angle results

$$\alpha_0=-y\frac{2}{\pi}$$

For $n=2.5$ the maximum is at 5/8 of the chord. The angle results $-\frac{y}{\pi}\frac{8}{3}$. The maximal height is 1.14 y.

(e) A section with a positive and negative curvature is

$$\xi=-y(1+x)(1-x)x$$

the zero angle is $\alpha_0=-\dfrac{y}{\pi}\displaystyle\int_{-1}^{+1}x\sqrt{\frac{1+x}{1-x}}\,dx$

$$\alpha_0=+\frac{y}{2}$$

(f) An example which is important in actual applications is a section which coincides with the chord at the front half of the chord, the second half being represented by the equation

$$\xi=y(1-x^2)x^2$$

At $x=0$ the section has a horizontal tangent. The angle of the rear tangent is $2y$. The zero angle for this section is

$$\alpha_0 = -\frac{y}{\pi}\int_0^1 \frac{x^2\sqrt{1-x^2}}{1+x}dx$$

$$\alpha_0 = -\frac{y}{\pi}\left|\frac{1}{2}\text{ arc sin } x + \left(\frac{2}{3}+\frac{x}{2}+\frac{x^2}{3}\right)\sqrt{1-x^2}\right|_0^1$$

$$\alpha_0 = -y\left[\frac{1}{4}+\frac{2}{3\pi}\right] = -0.46\ y$$

If the section is given graphically or by a table, the zero angle can be determined numerically. For this numerical calculation the following table can be used:

TABLE 1.

Per cent of chord	2.5	5	10	15	20	25	30	35	40	45	50
Factor	6.46	2.38	0.841	0.458	0.314	0.232	0.181	0.150	0.128	0.112	0.102
Per cent of chord		55	60	65	70	75	80	85	90	95	97.5
Factor		0.091	0.085	0.081	0.078	0.076	0.078	0.082	0.093	0.120	0.054

Determine the ordinates of the section starting at the chord. If the curve runs beneath the chord the ordinate must be considered negative. The chord is supposed to be divided into 100 per cent, 0 per cent being at the trailing edge and 100 per cent at the leading edge. Now multiply each ordinate for the per cents given in the first line of Table 1 by the factor given in the second line and add all products obtained. Their sum multiplied by $-\frac{57.3}{\pi t}$ gives the angle of attack in degrees at which the lift is zero. The chord and the ordinates of course have to be measured in the same units.

V. THIN SECTIONS WITH UPPER AND LOWER BOUNDARIES.

If the section has some thickness so that the upper and lower boundaries no longer coincide, the coefficients A in equation (13) are no longer zero. The section can be imagined to be created by first drawing the line of connection of all points equidistant from the upper and lower boundaries, which I will call the central curve of the wing section. Afterwards the thickness, which is supposed to be small when compared with the chord, is created by adding equal distances on both sides of the central curve. The coefficients B of equation (13) depend only on the central curve, the coefficients A only on the added thickness. The integral for the zero angle containing the B's only and not the A's, it can be supposed that the section with but a small thickness has the same zero angle and lift as its central line.

However, two things are then to be considered. A_0 in equation (12) is now no longer zero, but has a small finite value. Hence the first factor in the bracket of (14) is somewhat greater than 1 and hence the lift is slightly increased. It is not necessary, however, to pay attention to this fact. The real lift is always slightly smaller than the theoretical lift and the result is likely to be better if this refinement is neglected.

Another more serious difficulty occurs at the transformed leading edge of the section. If the leading edge of the section is blunt, its transformation shows a picture as in Figure 4. The new curve is no longer approximately parallel to the circle at the leading edge, but has an irregularity. The assumption for the correctness of the transformation (14) holds true no longer.

This can be remedied, however, by adding area to the central curve all around the leading point. The corner in Figure 4 is then filled and the resulting curve is smooth again (Fig. 5). The integral (19) is not affected by this change, if the addition is symmetrical. The increase of the area of the new curve is small even when compared with the original increase. The depth of the small triangle is filled if the added distance in the direction of the chord is as great as the thickness of the section near the leading edge. Hence the rule arises to make the central curve of the wing section end in the center of curvature of the leading edge.[1] The zero angle for a thin section with two boundaries is calculated as before, substituting in integral (27) the coordinates of the central curve from the rear edge to the center of curvature of the leading

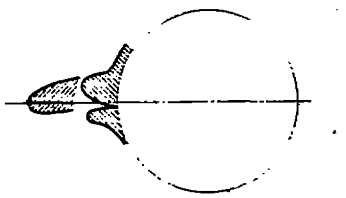


FIG. 4. Transformation of the head of a thick section.

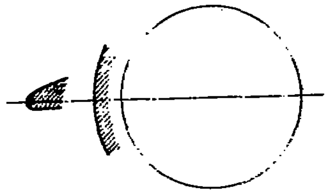


FIG. 5. Transformation of the head of a thick section, the central curve ending at the center of curvature.

edge. This central curve has to be brought to the standard length 2. The lift, however, is to be calculated as before, with the entire chord.

VI. THE MOMENT COEFFICIENT.

The previous discussion can easily be completed by the calculation of the moment which the thin section with small curvature experiences at small angles of attack. Remembering that the angle of attack α is a small quantity, the velocity of the flow along the circle, frequently mentioned before, is

$$2\,V_0(\sin\varphi-\alpha\cos\varphi+(\alpha-\alpha_0))$$

and the velocity at the corresponding points of the straight line is

$$V=V_0\left(1-\alpha\cot\varphi+\frac{\alpha-\alpha_0}{\sin\varphi}\right) \tag{28}$$

Only the term $-\alpha\cot\varphi$ contributes to the moment around the middle point $x=0$, and this moment for the chord 2 is

$$M_1=-\pi\rho\,\alpha\,V^2\qquad\frac{M_1}{t^2\,V^2\frac{\rho}{2}}=-\frac{\pi}{2}\alpha \tag{29}$$

After the variation of the chord into the section, the velocity at each point is only slightly changed in general. The variation is calculated by determining the variation of the potential. The corresponding point of the circle has no longer the same angle φ, but the angle φ is increased by a small quantity which may be denoted ϵ and is given by (14).

$$\epsilon=B_1\cos\varphi+B_2\cos 2\varphi++ \tag{31}$$

$$-A_1\sin\varphi-A_2\sin 2\varphi--$$

The variation of the potential is

$$2\epsilon V_0\sin\varphi$$

as a first approximation; and hence the variation of velocity at the points of the section is

$$-V_0\frac{\epsilon\cos\varphi+\frac{\alpha\epsilon}{\alpha\varphi}\sin\varphi}{\sin\varphi}$$

[1] A subsequent investigation shows that it is better to choose that point as end of the central curve which divides into two equal parts the distance between the center of curvature and the leading edge. I shall show this in a later paper.

The variation of the pressure is the product of density, original velocity, and variation of velocity. The moment is the integral of this variation of pressure, multiplied by $x\,dx = -4\cos\varphi\sin\varphi$ and therefore equals

$$M_2 = 8V_0^2\rho\int_0^{2\pi}\left(\frac{d\epsilon}{d\varphi}\cos\varphi\sin\varphi + \epsilon\cos^2\varphi\right)d\varphi \tag{32}$$

all quantities being omitted which contain the product of two small quantities.

Now

$$\frac{d\epsilon}{d\varphi} = \begin{array}{l} -B_1\sin\varphi - 2B_2\sin 2\varphi - - \\ -A_1\cos\varphi - 2A_2\cos 2\varphi - - \end{array}$$

Substituting ϵ and $\frac{d\epsilon}{d\varphi}$ only the terms with B contribute to the value of the integral, and I obtain

$$-M_2 = 4V^2\rho B_2\pi$$

B_2 was determined by

$$B_2 = -\frac{1}{\pi}\int_0^{2\pi}\frac{\xi}{2\sin\varphi}\sin 2\varphi d\varphi$$

or, expressed as a function of x,

$$B_2 = -\frac{1}{2\pi}\int_{-2}^{+2}\xi\frac{x}{(1-x/2)^{\frac{1}{2}}(1+x/2)^{\frac{1}{2}}}dx$$

Substituting this, I finally obtain for the moment

$$M_2 = 2V_0^2\rho\int_{-2}^{+2}\frac{x\ \xi dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \tag{33}$$

the corresponding absolute coefficient is

$$\frac{M_2}{S\,t\,V^2\frac{\rho}{2}} = \int_{-1}^{+1}\frac{x\xi dx}{\sqrt{1-x^2}} \text{ for the chord} = 2$$

The entire moment around the middle of the section is expressed by the absolute coefficient which is the sum of the two obtained coefficients

$$\frac{M}{S\,t\,V^2\frac{\rho}{2}} = -\alpha\frac{\pi}{2} + \int_{-1}^{+1}\frac{x\ \xi dx}{\sqrt{1-x^2}} \tag{35}$$

If the front and rear half of the central curve are equal, the integral in (35) is identically zero. The coefficient of moment then is

$$-\alpha\frac{\pi}{2}$$

This agrees with the result of Kutta for circular segments.

For sections with different upper and lower boundaries the central curve is to be taken again.

The numerical calculation of the integral in (35) can be performed by using the next table.

TABLE 2.

Per cent of chord	2.5	5	10	15	20	25	30	35	40	45	50
Factor	0.104	0.203	0.132	0.098	0.075	0.058	0.044	0.031	0.020	0.010	0.000

Per cent of chord	55	60	65	70	75	80	85	90	95	97.5
Factor	−0.010	−0.020	−0.031	−0.044	−0.058	−0.075	−0.098	−0.132	−0.203	−0.104

The procedure is analogue as with the calculation of the zero angle. The sum is to be multiplied by $\frac{2}{t}$ in order to obtain the additional coefficient of moment. The calculated coefficient of moment has not yet the same meaning as the moment coefficient C_m ordinarily used. This coefficient has reference to the leading edge and not to the middle of the section. Calling the integral in (35) the additional moment coefficient C_{m_0} the ordinary moment coefficient C_m results

$$C_m = -\alpha\frac{\pi}{2} + \int_{1-}^{+1} \frac{x\xi dx}{\sqrt{1-x^2}} + \frac{1}{2}C_L$$

(36) $$C_m = 0.25 \ C_L - \frac{\pi}{2}\alpha_0 + C_{m_0}$$

The center of pressure has the position

)37($$CP = \frac{C_m}{C_L} = 0.25 - \frac{\alpha_0\frac{\pi}{2} - C_{m_0}}{C_L}$$

VII. EXAMPLES OF THE MOMENT COEFFICIENT.

The additional moment coefficient is different from zero for unsymmetrical sections only and I proceed to calculate it for the two unsymmetrical sections discussed before.

This was first the section which in its front half was a straight line and in its rear half was represented by the equation

$$\xi = -y \ (1-x^2) \ x^2$$

The additional coefficient of moment C_m is

$$C_{m_0} = -y\int_0^1 \frac{(1-x^2)x^3}{\sqrt{1-x^2}}\,dx = -y\int_0^1 x^3\sqrt{1-x^2}\,dx$$

$$C_{m_0} = y \left| \frac{1}{3}(1-x^2)^{3/2} - \frac{1}{5}\,(1-x^2)^{5/2} \right|_0^1$$

$$= -\frac{2}{15}y.$$

As a second example take the S-shaped curve

$$\xi = -y \ (1-x^2)x$$

$$C_{m_0} = -y\int_{-1}^{+1} \frac{(1-x^2)x^2 \ dx}{\sqrt{1-x^2}}$$

$$C_{m_0} = -y\int_{-1}^{+1} x^2\sqrt{1-x^2} \ dx$$

$$C_{m_0} = -y \left| x(x^2 - \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}\text{ arc sin } x \right|_{-1}^{1+}$$

$$C_{m_0} = -\frac{\pi}{2}\,y$$

The values obtained have to be substituted in formula (36) or (37).

Equation (37) shows the center of pressure to be constant if $\frac{\pi}{2}\alpha_0 = C_{m_0}$; that is to say, if the moment coefficient is zero for zero lift. In most practical cases the right hand additional moment is too small to neutralize the moment at zero lift. The condition is fulfilled, however, for all sections with straight central curves, the moment to be neutralized being then zero too. These are not the only sections, however.

Theoretically, every tail of a section can be modified, so that the section has no longer any travel of the center of pressure, by superposing on the section the section discussed last but one, representing a bending upward of the tail.

A calculation shows, however, that this proceeding is not effective enough if the section was very unstable and does not lead then to good sections. Consider, for instance, the case of superposing a circular segment and this curve. The section may have the equation

$$\xi = y_1 (1-x^2) + y_2 (1-x^2)x^2$$

at the rear half, the front half is circular as before. The zero angle of this section is, according to the previous calculations

$$\alpha_0 = -y_1 + 0.64\, y_2$$

the additional moment is

$$-\frac{2}{15} y_2$$

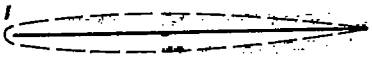
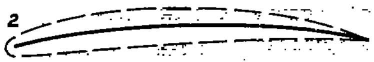
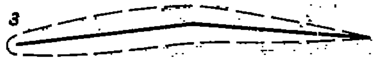
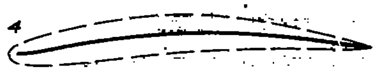
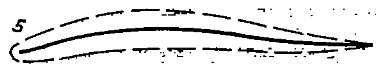
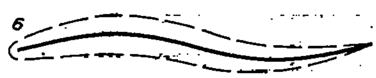
and the condition of constant center of pressure is

$$\frac{\pi}{2}\left[y_1 - 0.64\, y_2\right] = \frac{2}{15} y_2$$

It follows that

$$y_2 = .725\, y_1$$

TABLE 3.

Shape.	Equation of shape.	α_0	C_{m_0}
1 	$\xi=0$	0	0
2 	$\xi=y(1-x^2)$	$-y$	0
3 	$\xi=y(1\pm x)$	$-y\cdot\frac{2}{\pi}$	0
4 	$\xi=y(1-x^2)^{3/2}$	$-y\frac{2}{\pi}$	0
5 	$\xi=y(1-x)^{2.5}(1+x)^{1.5}$	$-y\frac{8}{3\pi}$	$+\frac{16}{15}y$
6 	$\xi=-y(1-x^2)x$	$-\frac{y}{2}$	$-\frac{\pi}{2}y$
7 	0 between -1 and 0 $\xi=yx^2(1-x^2)$ between 0 and $+1$	$y\left(\frac{1}{4}+\frac{2}{3\pi}\right)$	$-\frac{2}{15}y$

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